Problem 4.47

Because the three-dimensional harmonic oscillator potential (see Equation 4.215) is spherically symmetrical, the Schrödinger equation can also be handled by separation of variables in *spherical* coordinates. Use the power series method (as in Sections 2.3.2 and 4.2.1) to solve the radial equation. Find the recursion formula for the coefficients, and determine the allowed energies. (Check that your answer is consistent with Equation 4.216.) How is N related to n in this case? Draw the diagram analogous to Figures 4.3 and 4.6, and determine the degeneracy of nth energy level.[70]

[TYPO: Replace "of nth" with "of the nth."]

Solution

The governing equation for the wave function is Schrödinger's equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi + V\Psi$$

With a spherically symmetric potential energy function $V(r) = \frac{1}{2}M\omega^2 r^2$, the Laplacian operator can be expanded in spherical coordinates.

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2M}\nabla^2\Psi + V(r)\Psi(r,\theta,\phi,t) \\ &= -\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] + V(r)\Psi(r,\theta,\phi,t). \end{aligned}$$

The aim is to solve for $\Psi = \Psi(r,\theta,\phi,t)$ in all of space ($0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$) for $t > 0$. Assuming a product solution of the form $\Psi(r,\theta,\phi,t) = R(r)\Theta(\theta)\xi(\phi)T(t)$ and plugging it into the PDE yields the following system of ODEs (see Problem 4.4).

$$\left.\begin{aligned} i\hbar\frac{T'(t)}{T(t)} &= E \\ \frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right) - \frac{2Mr^2}{\hbar^2}[V(r)-E] &= F \\ \frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\Theta'(\theta)\sin\theta\right) + F\sin^2\theta &= G \\ -\frac{\xi''(\phi)}{\xi(\phi)} &= G \end{aligned}\right\}$$

The third and fourth eigenvalue problems are solved in Problem 4.4. The normalized products of angular eigenfunctions $\Theta(\theta)\xi(\phi)$ are called the spherical harmonics and are denoted by $Y_\ell^m(\theta,\phi)$. Solutions only exist if $F = \ell(\ell+1)$, where $\ell = 0, 1, 2, \ldots$, and if $G = m^2$ is an integer.

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,e^{im\phi}P_\ell^m(\cos\theta), \qquad \begin{cases} \ell = 0, 1, 2, \ldots \\ m = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell-1, \ell \end{cases}$$

[70]For some damn reason energy levels are traditionally counted starting with $n = 0$, for the harmonic oscillator. That conflicts with good sense and with our explicit convention (footnote 12), but please stick with it for this problem.

With these results the equation for $R(r)$ becomes

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right)-\frac{2Mr^2}{\hbar^2}[V(r)-E]=\ell(\ell+1)$$

$$\frac{d}{dr}\left[r^2\frac{dR}{dr}(r)\right]-\frac{2Mr^2}{\hbar^2}\left(\frac{1}{2}M\omega^2r^2-E\right)R(r)-\ell(\ell+1)R(r)=0.$$

Make the substitution,

$$R(r)=\frac{u(r)}{r}\quad\rightarrow\quad\frac{dR}{dr}=\frac{ru'-u}{r^2},$$

to eliminate the first derivative.

$$\frac{d}{dr}(ru'-u)-\frac{2Mr^2}{\hbar^2}\left(\frac{1}{2}M\omega^2r^2-E\right)\frac{u(r)}{r}-\ell(\ell+1)\frac{u(r)}{r}=0$$

$$r\frac{d^2u}{dr^2}+\cancel{\frac{du}{dr}}-\cancel{\frac{du}{dr}}-r\left(\frac{M^2\omega^2}{\hbar^2}r^2-\frac{2ME}{\hbar^2}\right)u(r)-\frac{\ell(\ell+1)}{r}u(r)=0$$

$$\frac{d^2u}{dr^2}-\left(\frac{M^2\omega^2}{\hbar^2}r^2-\frac{2ME}{\hbar^2}\right)u(r)-\frac{\ell(\ell+1)}{r^2}u(r)=0 \tag{1}$$

Make the following substitution to clean up the ODE.

$$r=\sqrt{\frac{\hbar}{M\omega}}\,\rho$$

Use the chain rule to find what the derivatives of u are in terms of this new variable.

$$\frac{du}{dr}=\frac{d\rho}{dr}\frac{du}{d\rho}=\sqrt{\frac{M\omega}{\hbar}}\frac{du}{d\rho}$$

$$\frac{d^2u}{dr^2}=\frac{d}{dr}\left(\frac{du}{dr}\right)=\frac{d\rho}{dr}\frac{d}{d\rho}\left(\sqrt{\frac{M\omega}{\hbar}}\frac{du}{d\rho}\right)=\sqrt{\frac{M\omega}{\hbar}}\frac{d}{d\rho}\left(\sqrt{\frac{M\omega}{\hbar}}\frac{du}{d\rho}\right)=\frac{M\omega}{\hbar}\frac{d^2u}{d\rho^2}$$

Then equation (1) becomes

$$\frac{M\omega}{\hbar}\frac{d^2u}{d\rho^2}-\left(\frac{M^2\omega^2}{\hbar^2}\cdot\frac{\hbar}{M\omega}\rho^2-\frac{2ME}{\hbar^2}\right)u(\rho)-\frac{\ell(\ell+1)}{\rho^2}\cdot\frac{M\omega}{\hbar}u(\rho)=0$$

$$\frac{d^2u}{d\rho^2}-\left(\rho^2-\frac{2E}{\hbar\omega}\right)u(\rho)-\frac{\ell(\ell+1)}{\rho^2}u(\rho)=0$$

$$\frac{d^2u}{d\rho^2}-\frac{\ell(\ell+1)}{\rho^2}u(\rho)+\left(\frac{2E}{\hbar\omega}-\rho^2\right)u(\rho)=0. \tag{2}$$

The aim now is to remove the asymptotic behavior from and hopefully simplify the ODE. If ρ is really small, then the second term dominates; in other words, the third term is negligible compared to the second term if $\rho\ll 1$.

$$\rho\ll 1:\quad\frac{d^2u}{d\rho^2}-\frac{\ell(\ell+1)}{\rho^2}u(\rho)=0\quad\Rightarrow\quad u(\rho)=C_1\rho^{\ell+1}+D_1\rho^{-\ell}$$

For the solution not to blow up, set $D_1 = 0$. The next substitution is apparent.

$$u(\rho) = \rho^{\ell+1}v(\rho)$$

Equation (2) becomes

$$\frac{d^2}{d\rho^2}[\rho^{\ell+1}v(\rho)] - \frac{\ell(\ell+1)}{\rho^2}[\rho^{\ell+1}v(\rho)] + \left(\frac{2E}{\hbar\omega} - \rho^2\right)[\rho^{\ell+1}v(\rho)] = 0$$

$$\rho^{\ell+1}\left[\frac{d^2v}{d\rho^2} + \frac{2}{\rho}(\ell+1)\frac{dv}{d\rho} + \cancel{\frac{\ell(\ell+1)}{\rho^2}v(\rho)}\right] - \cancel{\frac{\ell(\ell+1)}{\rho^2}[\rho^{\ell+1}v(\rho)]} + \left(\frac{2E}{\hbar\omega} - \rho^2\right)[\rho^{\ell+1}v(\rho)] = 0$$

$$\frac{d^2v}{d\rho^2} + \frac{2}{\rho}(\ell+1)\frac{dv}{d\rho} + \left(\frac{2E}{\hbar\omega} - \rho^2\right)v(\rho) = 0. \tag{3}$$

If ρ is really large, on the other hand, then the second term is negligible.

$$\rho \gg 1: \quad \frac{d^2v}{d\rho^2} + \left(\frac{2E}{\hbar\omega} - \rho^2\right)v(\rho) = 0$$

This is the same ODE encountered in the harmonic oscillator analysis on page 48. The fraction $2E/(\hbar\omega)$ is also negligible compared to ρ^2, leading to the approximate solution $v(\rho) = C_2e^{-\rho^2/2} + D_2e^{\rho^2/2}$. For the solution not to blow up, set $D_2 = 0$. The next substitution is apparent.

$$v(\rho) = e^{-\rho^2/2}w(\rho)$$

Equation (3) becomes

$$\frac{d^2}{d\rho^2}[e^{-\rho^2/2}w(\rho)] + \frac{2}{\rho}(\ell+1)\frac{d}{d\rho}[e^{-\rho^2/2}w(\rho)] + \left(\frac{2E}{\hbar\omega} - \rho^2\right)[e^{-\rho^2/2}w(\rho)] = 0$$

$$\{e^{-\rho^2/2}[w'' - 2\rho w' + (\rho^2 - 1)w]\} + \frac{2}{\rho}(\ell+1)[e^{-\rho^2/2}(w' - \rho w)] + \left(\frac{2E}{\hbar\omega} - \rho^2\right)[e^{-\rho^2/2}w(\rho)] = 0$$

$$\frac{d^2w}{d\rho^2} + \left[\frac{2}{\rho}(\ell+1) - 2\rho\right]\frac{dw}{d\rho} + \left[\frac{2E}{\hbar\omega} - 1 - 2(\ell+1)\right]w(\rho) = 0$$

$$\rho\frac{d^2w}{d\rho^2} + 2[(\ell+1) - \rho^2]\frac{dw}{d\rho} + (\varepsilon - 2\ell - 3)\rho w(\rho) = 0, \quad 0 < \rho < \infty, \tag{4}$$

where $\varepsilon = 2E/(\hbar\omega)$. A series solution to this ODE is sought with respect to $\rho = 0$. Since $\rho = 0$ is a regular singular point, it's mathematically proper to use a Frobenius series rather than a Taylor series.

$$w(\rho) = \rho^\gamma\sum_{j=0}^\infty c_j\rho^j = \sum_{j=0}^\infty c_j\rho^{j+\gamma}, \quad c_0 \neq 0$$

$$\frac{dw}{d\rho} = \sum_{j=0}^\infty c_j(j+\gamma)\rho^{j+\gamma-1}$$

$$\frac{d^2w}{d\rho^2} = \sum_{j=0}^\infty c_j(j+\gamma)(j+\gamma-1)\rho^{j+\gamma-2}$$

Substitute these formulas into equation (4).

$$
\begin{aligned}
0 &= \rho\sum_{j=0}^{\infty} c_j(j+\gamma)(j+\gamma-1)\rho^{j+\gamma-2} + 2[(\ell+1)-\rho^2]\sum_{j=0}^{\infty} c_j(j+\gamma)\rho^{j+\gamma-1} + (\varepsilon-2\ell-3)\rho\sum_{j=0}^{\infty} c_j\rho^{j+\gamma} \\
&= \sum_{j=0}^{\infty} c_j(j+\gamma)(j+\gamma-1)\rho^{j+\gamma-1} + 2[(\ell+1)-\rho^2]\sum_{j=0}^{\infty} c_j(j+\gamma)\rho^{j+\gamma-1} + (\varepsilon-2\ell-3)\sum_{j=0}^{\infty} c_j\rho^{j+\gamma+1} \\
&= \sum_{j=0}^{\infty} c_j(j+\gamma)(j+\gamma-1)\rho^{j+\gamma-1} + 2(\ell+1)\sum_{j=0}^{\infty} c_j(j+\gamma)\rho^{j+\gamma-1} \\
&\qquad - 2\sum_{j=0}^{\infty} c_j(j+\gamma)\rho^{j+\gamma+1} + (\varepsilon-2\ell-3)\sum_{j=0}^{\infty} c_j\rho^{j+\gamma+1} \\
&= c_0(\gamma)(\gamma-1)\rho^{\gamma-1} + c_1(\gamma+1)(\gamma)\rho^{\gamma} + 2(\ell+1)[c_0(\gamma)\rho^{\gamma-1} + c_1(\gamma+1)\rho^{\gamma}] \\
&\qquad + \sum_{j=2}^{\infty} c_j(j+\gamma)(j+\gamma-1)\rho^{j+\gamma-1} + 2(\ell+1)\sum_{j=2}^{\infty} c_j(j+\gamma)\rho^{j+\gamma-1} \\
&\qquad\qquad - 2\sum_{j=0}^{\infty} c_j(j+\gamma)\rho^{j+\gamma+1} + (\varepsilon-2\ell-3)\sum_{j=0}^{\infty} c_j\rho^{j+\gamma+1}
\end{aligned}
$$

Factor the terms in the first line in powers of ρ, make the substitution $j = k+2$ in the sums that start from 2, and make the substitution $j = k$ in the sums that start from 0.

$$
\begin{aligned}
0 &= c_0\gamma[(\gamma-1)+2(\ell+1)]\rho^{\gamma-1} + c_1(\gamma+1)[\gamma+2(\ell+1)]\rho^{\gamma} \\
&\qquad + \sum_{k+2=2}^{\infty} c_{k+2}(k+2+\gamma)(k+\gamma+1)\rho^{k+\gamma+1} + 2(\ell+1)\sum_{k+2=2}^{\infty} c_{k+2}(k+2+\gamma)\rho^{k+\gamma+1} \\
&\qquad\qquad - 2\sum_{k=0}^{\infty} c_k(k+\gamma)\rho^{k+\gamma+1} + (\varepsilon-2\ell-3)\sum_{k=0}^{\infty} c_k\rho^{k+\gamma+1} \\
&= c_0\gamma(\gamma+2\ell+1)\rho^{\gamma-1} + c_1(\gamma+1)(\gamma+2\ell+2)\rho^{\gamma} \\
&\qquad + \sum_{k=0}^{\infty}\left[c_{k+2}(k+2+\gamma)(k+\gamma+1) + 2(\ell+1)c_{k+2}(k+2+\gamma) - 2c_k(k+\gamma) + (\varepsilon-2\ell-3)c_k\right]\rho^{k+\gamma+1} \\
&= c_0\gamma(\gamma+2\ell+1)\rho^{\gamma-1} + c_1(\gamma+1)(\gamma+2\ell+2)\rho^{\gamma} \\
&\qquad + \sum_{k=0}^{\infty}\left\{c_{k+2}(k+2+\gamma)[(k+\gamma+1)+2(\ell+1)] - c_k[2(k+\gamma)-(\varepsilon-2\ell-3)]\right\}\rho^{k+\gamma+1} \\
&= c_0\gamma(\gamma+2\ell+1)\rho^{\gamma-1} + c_1(\gamma+1)(\gamma+2\ell+2)\rho^{\gamma} \\
&\qquad + \sum_{k=0}^{\infty}\left[c_{k+2}(k+2+\gamma)(k+\gamma+2\ell+3) - c_k(2k+2\gamma-\varepsilon+2\ell+3)\right]\rho^{k+\gamma+1}
\end{aligned}
$$

Match the coefficients of the powers of ρ on both sides.

$$\left.\begin{aligned} c_0\gamma(\gamma+2\ell+1) &= 0 \\ c_1(\gamma+1)(\gamma+2\ell+2) &= 0 \\ c_{k+2}(k+2+\gamma)(k+\gamma+2\ell+3) - c_k(2k+2\gamma-\varepsilon+2\ell+3) &= 0 \end{aligned}\right\}$$

Since c_0 is assumed to be the first nonzero coefficient of the series, divide both sides of the first equation by c_0.

$$\gamma(\gamma+2\ell+1) = 0$$

$$\gamma = 0 \quad \text{or} \quad \gamma = -1-2\ell$$

$\gamma = -1-2\ell$ is physically irrelevant because the first term in the series for $w(\rho)$ is $c_0\rho^\gamma$, and $w(\rho)$ has to be finite as $\rho \to 0$ and $\rho \to \infty$. The values of γ obtained by solving the second equation are irrelevant not only because they're negative, but also because the first equation can't be satisfied with $c_0 \neq 0$. Setting $\gamma = 0$ in the third equation gives

$$c_{k+2}(k+2)(k+2\ell+3) - c_k(2k-\varepsilon+2\ell+3) = 0$$

$$c_{k+2}(k+2)(k+2\ell+3) = (2k-\varepsilon+2\ell+3)c_k$$

$$\boxed{c_{k+2} = \frac{2k-\varepsilon+2\ell+3}{(k+2)(k+2\ell+3)}c_k, \quad k \geq 0}$$

for the recursion formula. Setting $\gamma = 0$ in the second equation gives

$$c_1(1)(2\ell+2) = 0 \quad \Rightarrow \quad c_1 = 0,$$

which means all odd coefficients (c_1, c_3, c_5, ...) are zero. Consider what happens now when k is large—the terms that do not have k are negligibly small.

$$c_{k+2} \approx \frac{2k}{(k)(k)}c_k = \left(\frac{2}{k}\right)c_k, \quad k \gg 1$$

Consequently,

$$\begin{aligned} c_{k+4} &= c_{(k+2)+2} \approx \left(\frac{2}{k+2}\right)c_{k+2} \approx \left(\frac{2}{k+2}\right)\left(\frac{2}{k}\right)c_k \\ c_{k+6} &= c_{(k+4)+2} \approx \left(\frac{2}{k+4}\right)c_{k+4} \approx \left(\frac{2}{k+4}\right)\left(\frac{2}{k+2}\right)\left(\frac{2}{k}\right)c_k \\ c_{k+8} &= c_{(k+6)+2} \approx \left(\frac{2}{k+6}\right)c_{k+6} \approx \left(\frac{2}{k+6}\right)\left(\frac{2}{k+4}\right)\left(\frac{2}{k+2}\right)\left(\frac{2}{k}\right)c_k \\ &\vdots \end{aligned}$$

for $k \gg 1$.

The series solution is then

$$\begin{aligned}
w(\rho) &= \rho^0 \sum_{j=0}^{\infty} c_j \rho^j \\
&= c_0 + \cancel{c_1\rho} + c_2\rho^2 + \cancel{c_3\rho^3} + c_4\rho^4 + \cdots + c_k\rho^k + \cancel{c_{k+1}\rho^{k+1}} + c_{k+2}\rho^{k+2} + \cdots \\
&= c_0 + c_2\rho^2 + c_4\rho^4 + \cdots + c_k\rho^k + c_{k+2}\rho^{k+2} + c_{k+4}\rho^{k+4} + c_{k+6}\rho^{k+6} + c_{k+8}\rho^{k+8} + \cdots \\
&\approx c_0 + c_2\rho^2 + c_4\rho^4 + \cdots + c_k\rho^k + \left(\frac{2}{k}\right) c_k\rho^{k+2} + \left(\frac{2}{k+2}\right)\left(\frac{2}{k}\right) c_k\rho^{k+4} \\
&\qquad + \left(\frac{2}{k+4}\right)\left(\frac{2}{k+2}\right)\left(\frac{2}{k}\right) c_k\rho^{k+6} + \left(\frac{2}{k+6}\right)\left(\frac{2}{k+4}\right)\left(\frac{2}{k+2}\right)\left(\frac{2}{k}\right) c_k\rho^{k+8} + \cdots \\
&\approx c_0 + c_2\rho^2 + c_4\rho^4 + \cdots + c_k\rho^k + \left(\frac{1}{\frac{k}{2}}\right) c_k\rho^{k+2} + \left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{\frac{k}{2}}\right) c_k\rho^{k+4} \\
&\qquad + \left(\frac{1}{\frac{k}{2}+2}\right)\left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{\frac{k}{2}}\right) c_k\rho^{k+6} + \left(\frac{1}{\frac{k}{2}+3}\right)\left(\frac{1}{\frac{k}{2}+2}\right)\left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{\frac{k}{2}}\right) c_k\rho^{k+8} + \cdots \\
&\approx c_0 + c_2\rho^2 + c_4\rho^4 + \cdots + c_k\rho^k + c_k\rho^2\left(\frac{1}{\frac{k}{2}}\right)\rho^k + c_k\rho^4\left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{\frac{k}{2}}\right)\rho^k \\
&\qquad + c_k\rho^6\left(\frac{1}{\frac{k}{2}+2}\right)\left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{\frac{k}{2}}\right)\rho^k + c_k\rho^8\left(\frac{1}{\frac{k}{2}+3}\right)\left(\frac{1}{\frac{k}{2}+2}\right)\left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{\frac{k}{2}}\right)\rho^k + \cdots .
\end{aligned}$$

Make the substitution $l = k/2$, or $k = 2l$.

$$\begin{aligned}
w(\rho) &\approx c_0 + c_2\rho^2 + c_4\rho^4 + \cdots + c_{2l}\rho^{2l} + c_{2l}\rho^2\left(\frac{1}{l}\right)\rho^{2l} + c_{2l}\rho^4\left(\frac{1}{l+1}\right)\left(\frac{1}{l}\right)\rho^{2l} \\
&\qquad + c_{2l}\rho^6\left(\frac{1}{l+2}\right)\left(\frac{1}{l+1}\right)\left(\frac{1}{l}\right)\rho^{2l} + c_{2l}\rho^8\left(\frac{1}{l+3}\right)\left(\frac{1}{l+2}\right)\left(\frac{1}{l+1}\right)\left(\frac{1}{l}\right)\rho^{2l} + \cdots \\
&\approx c_0 + c_2\rho^2 + c_4\rho^4 + \cdots + c_{2l}(\rho^2)^l + c_{2l}\rho^2\left(\frac{1}{l}\right)(\rho^2)^l + c_{2l}\rho^4\left(\frac{1}{l+1}\right)\left(\frac{1}{l}\right)(\rho^2)^l \\
&\qquad + c_{2l}\rho^6\left(\frac{1}{l+2}\right)\left(\frac{1}{l+1}\right)\left(\frac{1}{l}\right)(\rho^2)^l + c_{2l}\rho^8\left(\frac{1}{l+3}\right)\left(\frac{1}{l+2}\right)\left(\frac{1}{l+1}\right)\left(\frac{1}{l}\right)(\rho^2)^l + \cdots
\end{aligned}$$

Based on the form of these terms in the sum involving l, the most rapidly changing component of the leading behavior of $w(\rho)$ as $\rho \to \infty$ is suspected to be e^{ρ^2}. To show this, notice from the ODE for $w(\rho)$ in equation (4) that ∞ is an irregular singular point, and apply the method of dominant balance to determine how $w(\rho)$ blows up as $w \to \infty$. Start by making the standard substitution $w(\rho) = e^{S(\rho)}$ in equation (4); use the chain rule to write formulas for the derivatives in terms of this new variable.

$$\frac{dw}{d\rho} = e^{S(\rho)} S'$$

$$\frac{d^2w}{d\rho^2} = e^{S(\rho)}(S')^2 + e^{S(\rho)}S'' = e^{S(\rho)}[(S')^2 + S'']$$

As a result, equation (4) becomes

$$\rho e^{S(\rho)}[(S')^2 + S''] + 2[(\ell+1) - \rho^2]e^{S(\rho)}S' + (\varepsilon - 2\ell - 3)\rho e^{S(\rho)} = 0.$$

Divide both sides by $e^{S(\rho)}$.

$$\rho[(S')^2 + S''] + 2[(\ell+1) - \rho^2]S' + (\varepsilon - 2\ell - 3)\rho = 0$$

In the limit as $\rho \to \infty$, S' and S'' are assumed to have the same order of magnitude, so $(S')^2 \gg S''$ and $\rho^2 \gg (\ell+1)$.

$$\rho \to \infty : \quad \rho(S')^2 - 2\rho^2 S' \sim (2\ell + 3 - \varepsilon)\rho$$

Solve this asymptotic differential equation for S.

$$(S')^2 \sim 2\rho S' + (2\ell + 3 - \varepsilon)$$

$$S' \sim \frac{2\rho \pm \sqrt{4\rho^2 + 4(2\ell + 3 - \varepsilon)}}{2}$$

$4(2\ell + 3 - \varepsilon)$ is negligible compared to $4\rho^2$.

$$S' \sim \frac{2\rho \pm \sqrt{4\rho^2}}{2}$$

$$S' \sim 2\rho$$

$$S \sim \rho^2$$

Fortunately, $S(\rho) \sim \rho^2$ as $\rho \to \infty$ is consistent with the assumption that $(S')^2 \gg S''$ as $\rho \to \infty$, so no further analysis is needed; the most rapidly changing component of the leading behavior of $w(\rho)$ as $\rho \to \infty$ is in fact e^{ρ^2}. Then

$$v(\rho) = e^{-\rho^2/2}w(\rho)$$

increases as $e^{\rho^2/2}$, which blows up as $\rho \to \infty$. For there to be a physically realistic solution, the series solution for $w(\rho)$ must terminate at some maximum even value of the index. This can only happen if the numerator of the boxed recursion relation is zero for some even value of k: $k_{\text{max}} = 2N$, where $N = 0, 1, 2, \ldots$.

$$2k_{\text{max}} - \varepsilon + 2\ell + 3 = 0$$

$$\varepsilon = 2k_{\text{max}} + 2\ell + 3$$

$$\frac{2E}{\hbar\omega} = 2(2N) + 2\ell + 3$$

$$E = \hbar\omega\left(2N + \ell + \frac{3}{2}\right)$$

Therefore, using a new constant $\boxed{n = 2N + \ell,}$

$$\boxed{E_n = \hbar\omega\left(n + \frac{3}{2}\right), \quad n = 0, 1, 2, \ldots.}$$

As Mr. Griffiths notes, n starts at zero for the harmonic oscillator; this is because both N and ℓ start from zero. For a given ℓ, there are multiple stationary states with the same energy—one with $m = -\ell$, another with $m = -\ell + 1$, and so on until $m = \ell$. Recall that degenerate states (DSs) are states that have the same energy.

If $n = 0$, then $(\underbrace{N = 0 \text{ and } \ell = 0}_{1})$, meaning there is one degenerate state—the ground state.

If $n = 1$, then $(\underbrace{N = 0 \text{ and } \ell = 1}_{3})$, meaning there are three DSs.

If $n = 2$, then $(\underbrace{N = 0 \text{ and } \ell = 2}_{5})$ or $(\underbrace{N = 1 \text{ and } \ell = 0}_{1})$, meaning there are six DSs.

If $n = 3$, then $(\underbrace{N = 0 \text{ and } \ell = 3}_{7})$ or $(\underbrace{N = 1 \text{ and } \ell = 1}_{3})$, meaning there are ten DSs.

If $n = 4$, then $(\underbrace{N = 0 \text{ and } \ell = 4}_{9})$ or $(\underbrace{N = 1 \text{ and } \ell = 2}_{5})$ or $(\underbrace{N = 2 \text{ and } \ell = 0}_{1})$, meaning there are 15 DSs.

If $n = 5$, then $(\underbrace{N = 0 \text{ and } \ell = 5}_{11})$ or $(\underbrace{N = 1 \text{ and } \ell = 3}_{7})$ or $(\underbrace{N = 2 \text{ and } \ell = 1}_{3})$, meaning there are 21 DSs.

As a result, $d_0 = 1$, $d_1 = 3$, $d_2 = 6$, and $d_3 = 10$. Notice that to get d_1, 2 needs to be added to d_0; to get d_2, 3 needs to be added to d_1; and to get d_3, 4 needs to be added to d_2. The pattern is apparent for d_{n+1}.

$$d_{n+1} = (n + 2) + d_n, \quad d_0 = 1$$

This is a recurrence relation, more specifically an inhomogeneous first-order linear difference equation with constant coefficients. Bring d_n to the left side.

$$d_{n+1} - d_n = n + 2$$

The left side is how the discrete derivative of a function d_n of the integers is defined.

$$Dd_n = n + 2$$

Take the discrete antiderivative of both sides by summing from 0 to $n - 1$.

$$\sum_{q=0}^{n-1} Dd_q = \sum_{q=0}^{n-1} (q + 2)$$

$$Dd_0 + Dd_1 + Dd_2 + \cdots + Dd_{n-2} + Dd_{n-1} = \sum_{q=0}^{n-1} q + \sum_{q=0}^{n-1} 2$$

$$(d_1 - d_0) + (d_2 - d_1) + \cdots + (d_{n-1} - d_{n-2}) + (d_n - d_{n-1}) = 0 + \sum_{q=1}^{n-1} q + 2\sum_{q=0}^{n-1} 1$$

$$d_n - d_0 = \frac{(n-1)[(n-1)+1]}{2} + 2[(n-1)+1]$$

Simplify the right side and solve for d_n.

$$d_n - 1 = \frac{(n-1)n}{2} + 2n$$

$$d_n - 1 = \frac{n^2 + 3n}{2}$$

$$d_n = \frac{n^2 + 3n + 2}{2}$$

Therefore, the degeneracy of energy E_n is

$$\boxed{d_n = \frac{(n+2)(n+1)}{2}}.$$

Below is an energy-level diagram.